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The influence of surface tension on cavitating flow past a curved obstacle

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The problem of cavitating flow past a two-dimensional curved obstacle is considered. Surface tension is included in the dynamic boundary condition. A perturbation solution for small values of the surface tension is presented. It is found that the position of the separation points is uniquely determined by specifying the value of the Weber number and the contact angle at the separation points. In addition, for a given value of the Weber number there exists a particular position of the separation points for which the slope is continuous. This solution tends to the classical solution satisfying the Brillouin–Villat condition as the surface tension tends to zero. Graphs of the results for the cavitating flow past a circular cylinder are presented.

1. Introduction

In recent years important progress has been achieved in the understanding of the influence of surface tension on cavitating flow past a flat plate. The classical Helmholtz–Kirchoff solution yields infinite curvature of the free surface at the edges of the plate. Ackerberg (1975) constructed an asymptotic solution for small values of the surface tension in which the slope and the curvature of the free surface at the edges are both equal to those of the plate. Ackerberg's solution contains capillary waves downstream. Cumberbatch & Norbury (1979) observed that these waves are not physically acceptable because they require a supply of energy from infinity. They suggested that solutions without waves could be obtained by forcing the slope of the free surface at the edges to be equal to the slope of the plate and allowing the curvature to be different from zero at the edges. Although they obtained a local solution, they did not succeed in matching it with any outer solution. The problem was solved by Vanden-Broeck (1981), who provided conclusive analytical and numerical evidence that the slope is not continuous at the separation points. Both velocity and curvature are infinite there.

In the present paper we generalize Vanden-Broeck's results to the cavitating flow past a curved obstacle (see figure 1). The position of the separation point may be either fixed if it is at a pointed corner of the body, or free if it is at a certain location of a smoothly curved obstacle. An example of fixed detachment is provided by the cavitating flow past a flat plate in which the flow leaves the plate at the edges. Similarly the flow sketched in figure 1 corresponds to fixed detachment if the obstacle is cut along the straight line AB. In the case of free detachment, the classical solution leaves the position of the separation points A and B undetermined. This degeneracy is usually resolved by imposing the Brillouin–Villat condition, which requires the curvature of the free surface to be finite at the separation points (Birkhoff & Zarantonello 1957).

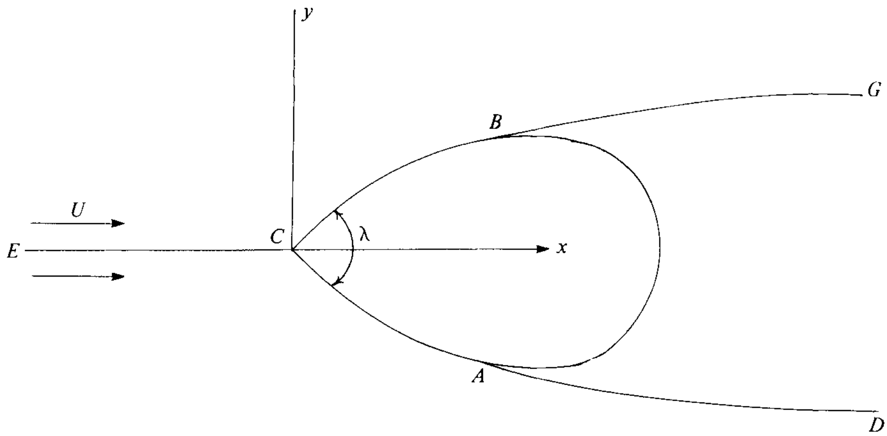

FIGURE 1. Sketch of the flow and the coordinates.

The problem is formulated in §2, and the classical solution without surface tension is computed numerically in §3. The scheme is similar in philosophy if not in details to the scheme derived by Brodetsky (1923) and later extended by Birkhoff, Goldstein & Zarantonello (1953) and Birkhoff, Young & Zarantonello (1954). Explicit results are presented for the cavitating flow past a circular cylinder.

In §4 the numerical solution of §3 is used to construct an asymptotic solution for small values of the surface tension. It is found that, for most positions of the separation points, the slope is not continuous at A and B. The velocity is infinite or equal to zero there. The position of the separation points is uniquely determined by specifying the value of the Weber number and the contact angle at the separation points. In addition, for a given value of the Weber number there exists a particular position of the separation points A and B for which the slope is continuous at A and B. This solution tends to the classical solution satisfying the Brillouin–Villat condition as the surface tension tends to zero.

2. Formulation

We consider the cavitating flow past a curved obstacle (see figure 1). We denote by L a typical dimension of the obstacle. At infinity we have flow with constant velocity U. The fluid is assumed to be inviscid and incompressible. We restrict our attention to obstacles which are symmetrical with respect to the direction of the velocity at infinity. Flows past non-symmetrical obstacles can be treated similarly. It is convenient to introduce dimensionless variables by choosing L as the unit length and U as the unit velocity.

We introduce the dimensionless potential ϕb and stream function ψb. The constant b is chosen such that $\phi = 1$ at the separation points. Without loss of generality we choose $\phi = 0$ at $x = y = 0$. The free surface, the obstacle and the negative x-axis are portions of the streamline $\psi = 0$.

We denote the complex velocity by $u - \mathrm{i}v$ and we define the function $\tau - \mathrm{i}\theta$ by the relation

$$u - \mathrm{i}v = \mathrm{e}^{\tau - \mathrm{i}\theta}. \tag{2.1}$$

We shall seek $\tau - \mathrm{i}\theta$ as an analytic function of $f = \phi + \mathrm{i}\psi$ in the half-plane $\psi \leqslant 0$. The complex potential plane is sketched in figure 2. At infinity we require the velocity

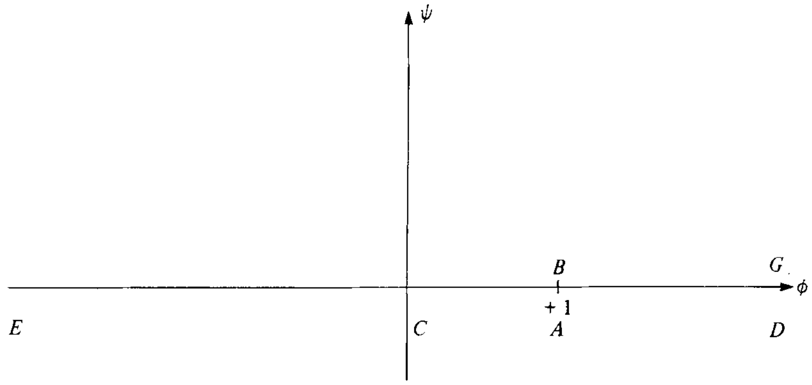

FIGURE 2. The image of the flow in the plane of the complex potential $f = \phi + i\psi$.

to be unity in the x-direction so that the function $\tau - i\theta$ vanishes at infinity in view of (2.1).

On the surface of the cavity the Bernoulli equation and the pressure jump due to surface tension yield

$$\tfrac{1}{2}q^2 - \frac{T}{\rho}K = \tfrac{1}{2}U^2. \tag{2.2}$$

Here q is the flow speed, K the curvature of the cavity surface counted positive when the centre of curvature lies inside the fluid regions, T the surface tension and ρ the density. In dimensionless variables this becomes (for details see Ackerberg 1975)

$$\frac{e^{\tau}}{b}\frac{\partial\theta}{\partial\phi} = \tfrac{1}{2}\alpha(e^{2\tau}-1) \quad (1 < \phi < \infty). \tag{2.3}$$

Here α is the Weber number defined by

$$\alpha = \frac{\rho U^2 L}{T}. \tag{2.4}$$

The symmetry of the problem and the kinematic condition on the obstacle yield

$$\theta(\phi) = 0 \quad (\psi = 0,\ \phi < 0) \tag{2.5}$$

$$F[x(\phi), y(\phi)] = 0 \quad (\psi = 0,\ 0 < \phi < 1). \tag{2.6}$$

Here $F(x, y) = 0$ is the equation of the shape of the obstacle, and the functions $\theta(\phi)$, $x(\phi)$ and $y(\phi)$ denote respectively $\theta(\phi, 0-)$, $x(\phi, 0-)$ and $y(\phi, 0-)$.

This completes the formulation of the problem of determining the function $\tau - i\theta$ and the constant b. For each value of α, $\tau - i\theta$ must be analytic in the half-plane $\psi \leqslant 0$ and satisfy the boundary conditions (2.3), (2.5) and (2.6).

3. Solution without surface tension

When surface tension is neglected, the Weber number is infinite and the condition (2.3) reduces to the free-streamline condition $\tau = 0$.

We define the new variable t by the transformation

$$f^{\frac{1}{2}} = \left(t - \frac{1}{t}\right)\frac{1}{2i}. \tag{3.1}$$

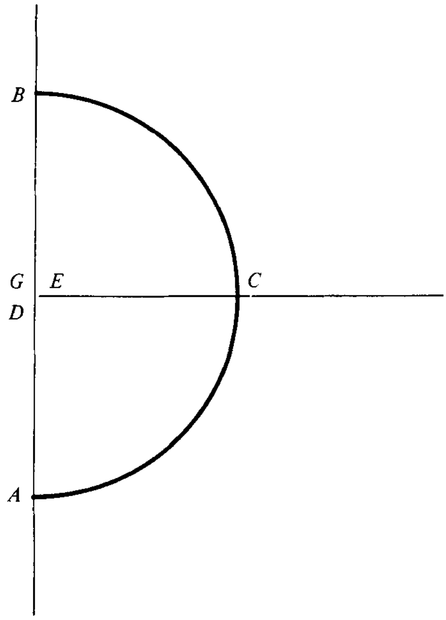

FIGURE 3. The t-plane.

The problem in the complex plane t is illustrated in figure 3. Following Brodetsky (1923) we introduce the function $\Omega'(t)$ by the relation

$$\tau - \mathrm{i}\theta = -\frac{\lambda}{\pi}\log\frac{1+t}{1-t} - \Omega'(t), \tag{3.2}$$

where the angle λ is defined in figure 1. The conditions (2.3) and (2.5) show that $\Omega'(t)$ can be expressed in the form of a Taylor expansion in odd powers of t. Hence

$$\tau - \mathrm{i}\theta = -\frac{\lambda}{\pi}\log\frac{1+t}{1-t} - \sum_{n=1}^{\infty} A_n t^{2n-1}. \tag{3.3}$$

The function (3.3) satisfy the conditions (2.3) and (2.5). The coefficients A_n have to be determined to satisfy the condition (2.6) on the surface ACB of the obstacle. We use the notation $t = r\,\mathrm{e}^{\mathrm{i}\sigma}$ so that points on ACB are given by $r = 1$, $-\frac{1}{2}\pi \leqslant \sigma \leqslant \frac{1}{2}\pi$. Using (3.1) and (2.1) we have

$$\frac{\partial x}{\partial \sigma} = b \sin 2\sigma\, \mathrm{e}^{-\tau} \cos\theta \quad (\rho = 1,\ -\tfrac{1}{2}\pi \leqslant \sigma \leqslant \tfrac{1}{2}\pi), \tag{3.4}$$

$$\frac{\partial y}{\partial \sigma} = b \sin 2\sigma\, \mathrm{e}^{-\tau} \sin\theta \quad (\rho = 1,\ -\tfrac{1}{2}\pi \leqslant \sigma \leqslant \tfrac{1}{2}\pi). \tag{3.5}$$

We solve the problem approximately by truncating the infinite series in (3.3) after N terms. We find the N coefficients A_n and the constant b by a hybrid method involving collocation and finite differences. Substituting $t = \mathrm{e}^{\mathrm{i}\sigma}$ into (3.3) we have

$$\theta(\sigma) = -\tfrac{1}{2}\lambda + \sum_{n=1}^{N} A_n \sin[(2n-1)\sigma], \tag{3.6}$$

$$\tau(\sigma) = -\frac{\lambda}{\pi}\log\frac{-\sin\sigma}{1-\cos\sigma} - \sum_{n=1}^{N} A_n \cos[(2n-1)\sigma]. \tag{3.7}$$

We now introduce the N mesh points

$$\sigma_I = -\frac{\pi}{2N} I \quad (I = 1, \ldots, N) \tag{3.8}$$

and the N intermediate mesh points

$$\sigma_I^M = -\frac{\pi}{2N}(I-\tfrac{1}{2}) \quad (I = 1, \ldots, N). \tag{3.9}$$

Using (3.4)–(3.7) and (3.9) we obtain

$$\left(\frac{\partial x}{\partial \sigma}\right)_{\sigma = \sigma_I^M} \quad \text{and} \quad \left(\frac{\partial y}{\partial \sigma}\right)_{\sigma = \sigma_I^M}$$

in terms of the coefficients A_n and the constant b. These expressions enable us to evaluate $x(\sigma_I)$ and $y(\sigma_I)$ by the trapezoidal rule. Then (2.6) provides N algebraic equations for the $N+1$ unknowns A_n and b, namely

$$F[x(\sigma_I), y(\sigma_I)] = 0 \quad (I = 1, \ldots, N). \tag{3.10}$$

The last equation is obtained by specifying the abscissa w of the separation point A. Thus

$$x(-\tfrac{1}{2}\pi) = w. \tag{3.11}$$

The system (3.10), (3.11) is easily solved by Newton's method. Explicit computations were performed for the cavitating flow past a circular cylinder. The unit length L was chosen as the radius of the cylinder. The scheme converges rapidly and the solutions obtained were found to agree with the numerical results given by Birkhoff & Zarantonello (1957).

Profiles of the cavity for various values of the angular position γ of the separation points are presented in figure 4. For $\gamma < \gamma^* \approx 55°$ the free surface enters the body. These solutions are acceptable if the body is cut along the straight line AB. For $\gamma > \gamma^{**} \approx 124°$, the free surfaces cross over and the corresponding solutions are not physically acceptable. Physically acceptable solutions for $\gamma > \gamma^{**}$ can be obtained by using the method presented by Vanden-Broeck & Keller (1980) to prevent overlapping in capillary waves of large amplitude. These solutions are found to be the cusped cavities considered before by Southwell & Vaisey (1946), Lighthill (1949) and others (see figure 4). The pressure in the cavity is found as part of the solution. Similarly, in the work of Vanden-Broeck & Keller (1980) the pressure in the trapped bubble was found as part of the solution. As γ tends to γ^{**} the pressure in the cavity tends to zero. As γ tends to 180° the cavity shrinks to a point, and the solution reduces to the classical potential flow past a circle. Thus the family of cusped cavities is the physical continuation for $\gamma > \gamma^{**}$ of the family of open cavities.

The curvature of the free surface in the neighbourhood of the separation point A is given by the formula (Brodetsky 1923)

$$\frac{1}{b}\frac{\partial \theta}{\partial \phi} \sim -\tfrac{1}{2}C(b\phi - b)^{-\frac{1}{2}} \quad \text{as } \phi \to 1, \tag{3.12}$$

where

$$C = -\frac{\lambda b^{-\frac{1}{2}}}{\pi} - b^{-\frac{1}{2}} \sum_{n=1}^{N} (-1)^{n+1}(2n-1)A_n. \tag{3.13}$$

These formula are true for the cavitating flow past any curved obstacle.

A graph of C versus the angular position of γ of the separation points for the flow past a circular cylinder is shown in figure 5. The constant C vanishes for $\gamma = \gamma^*$. Thus

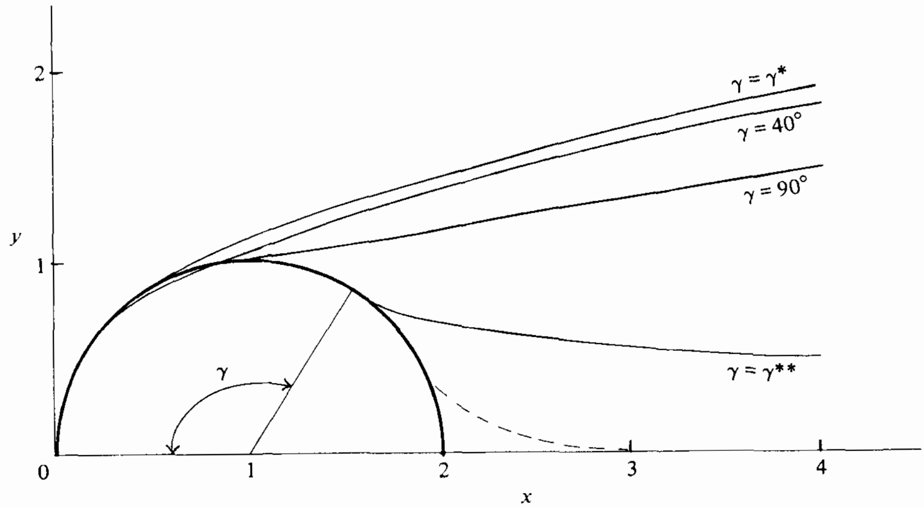

FIGURE 4. Cavities without surface tension in steady two-dimensional flow past a circular cylinder for $\gamma = 40°$, $\gamma = \gamma^* \approx 55°$, $\gamma = 90°$ and $\gamma = \gamma^{**} \approx 124°$. The broken line represents a cusped cavity computed numerically by Southwell & Vaisey (1946). The velocity on the free-streamlines of the cusped cavity is equal to $0.6U$. The corresponding cavitation number is equal to -0.64.

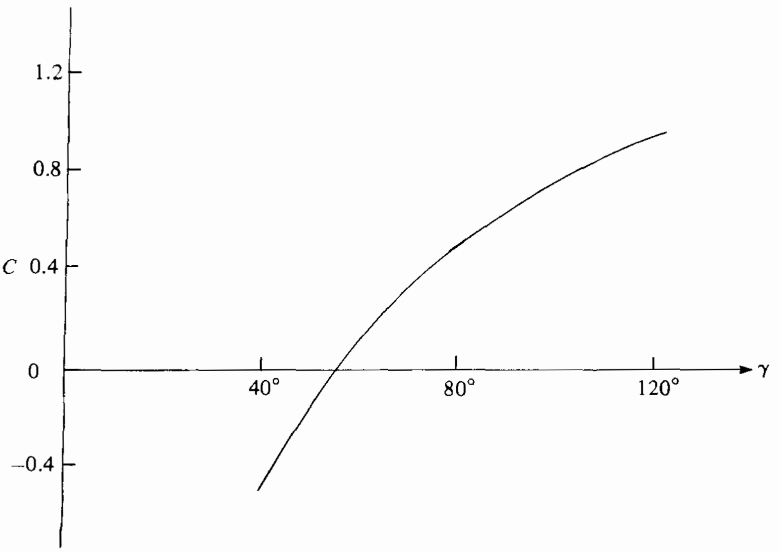

FIGURE 5. Computed values of the parameter C as a function of the angle γ.

(3.12) shows that the curvature at the separation points is infinite unless $\gamma = \gamma^*$. If we impose the Brillouin–Villat condition, the problem with free detachment has a unique solution corresponding to $\gamma = \gamma^*$. These numerical results are confirmed by the existence and uniqueness theorems mentioned by Birkhoff & Zarantonello (1957).

4. Perturbation solution for small values of the surface tension

We now assume that the Weber number α is very large. The solution for $\alpha = \infty$ is simply the solution without surface tension derived in § 3. Formula (3.12) indicates that the solution of § 3 cannot be uniformly valid as $\alpha \to \infty$, for the left-hand side of (2.3) can be made arbitrarily large if ϕ is chosen sufficiently close to unity. Thus we seek a solution in the vicinity of the separation point A. Following Ackerberg (1975) we introduce the following scaling of the variables:

$$f^* = \alpha(bf-b), \tag{4.1}$$

$$\tau^*-\mathrm{i}\theta^* = \alpha^{\frac{1}{2}}(\tau-\mathrm{i}\theta-\tfrac{1}{2}\mathrm{i}\pi+\mathrm{i}\gamma). \tag{4.2}$$

The function τ^* satisfies Laplace's equation in the lower half-plane $\psi^* \leqslant 0$. Thus

$$\frac{\partial^2\tau^*}{\partial\phi^{*2}}+\frac{\partial^2\tau^*}{\partial\psi^{*2}} = 0 \quad (\psi^* < 0). \tag{4.3}$$

The boundary conditions (2.3) and (2.6) linearize in the limit $\alpha\to\infty$, so that the boundary conditions on $\psi^* = 0$ are (for details see Ackerberg 1975)

$$\frac{\partial\tau^*}{\partial\psi^*} = 0 \quad (\psi^* = 0,\ \phi^* < 0) \tag{4.4}$$

$$\frac{\partial\tau^*}{\partial\psi^*} = \tau^* \quad (\psi^* = 0,\ \phi^* > 0). \tag{4.5}$$

Relation (3.12) gives the behaviour

$$\tau^* \sim \operatorname{Im} C(f^*)^{\frac{1}{2}} \quad \text{as} \quad |f^*|\to\infty. \tag{4.6}$$

Cumberbatch & Norbury (1979) noticed that the problem (4.3)–(4.5) had been treated by Friedrichs & Levy (1948). The solution of (4.3)–(4.6) not containing waves and having the weakest singularity at A is given on the free surface by

$$\theta^*(\phi^*) = \tfrac{1}{2}C\pi^{\frac{1}{2}}+\frac{C}{2\pi^{\frac{1}{2}}}\phi^*\ln\phi^* \quad \text{as} \quad \phi^*\to 0, \tag{4.7}$$

$$\tau^*(\phi^*) = \frac{1}{2\pi^{\frac{1}{2}}}C\ln\phi^* \quad \text{as} \quad \phi^*\to 0. \tag{4.8}$$

The leading-order terms in (4.7) and (4.8) correspond to flow past a corner of angle

$$\delta = \pi+\frac{C}{2}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}. \tag{4.9}$$

However, the solution (4.7), (4.8) is not valid near $\phi^* = 0$ because τ^* is unbounded at $\phi^* = 0$.

The asymptotic scheme can now be described as follows. For ϕ large we have the outer solution whose first term is given by the solution of §3. This solution merges into the solution of (4.3)–(4.6) obtained for $\phi^* \sim 1$, i.e. for $\phi-1 \sim \alpha^{-1}$. The solution of (4.3)–(4.6) becomes invalid for $\phi^* < 1$, i.e. for $\phi-1 < \alpha^{-1}$, because τ^* is unbounded at $\phi^* = 0$. In order to complete the perturbation calculation we have therefore to find a local solution valid for $\phi-1 < \alpha^{-1}$. Following Vanden-Broeck (1981) we seek a local solution which corresponds to a flow past a corner of angle δ. Thus we write

$$\mathrm{e}^{\tau} \sim E(\Phi-1)^{\pi/(2\pi-\delta)-1}. \tag{4.10}$$

Here E is a constant to be determined as part of the solution. Substitution (4.10) into (2.3) we have

$$\frac{\partial\theta}{\partial\phi} \sim \tfrac{1}{2}\alpha b\,\{E(\phi-1)^{\pi/(2\pi-\delta)-1}-E^{-1}(\phi-1)^{1-\pi/(2\pi-\delta)}\}. \tag{4.11}$$

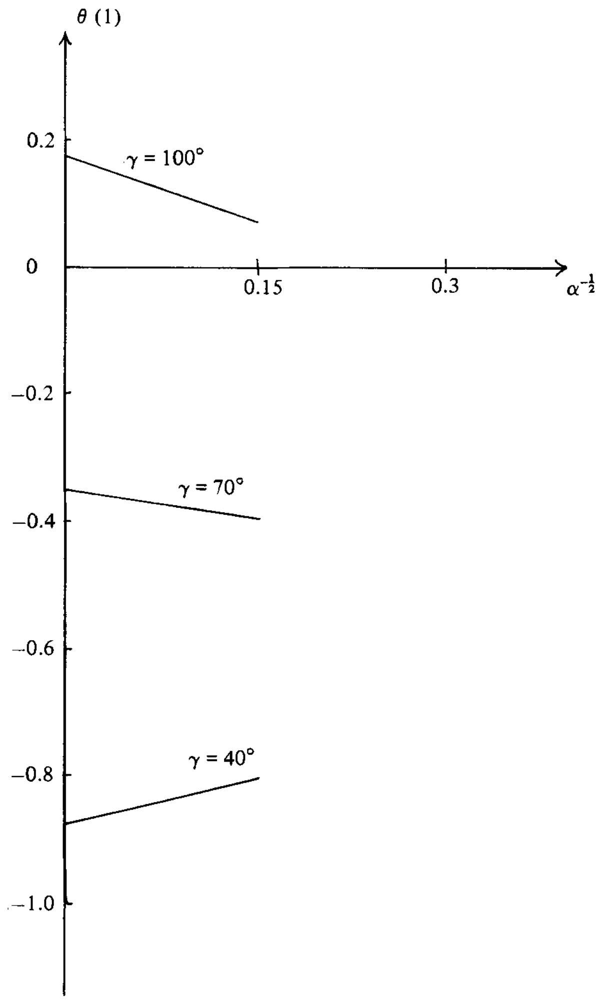

FIGURE 6. Computed values of $\theta(1)$ as a function of $\alpha^{-\frac{1}{2}}$ for $\gamma = 40°$, $70°$ and $100°$.

Matching (4.11) and (4.7) we find

$$E = 1. \tag{4.12}$$

Thus we have succeeded in matching the solution (4.7), (4.8) with a local solution corresponding to the flow past a corner of angle δ.

The value of θ at the separation point A is given by

$$\theta(1) = -\tfrac{1}{2}\pi + \gamma - \frac{C}{2}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}. \tag{4.13}$$

Relation (4.9) shows that $\delta > \pi$ for $C > 0$ and $\delta < \pi$ for $C < 0$. Therefore the velocity at the separation points is infinite for $C < 0$ and equal to zero for $C > 0$.

For the flow past a flat plate $C = -(\pi+4)^{\frac{1}{2}}$, and the formulas (4.7)–(4.9) reduce to the formulas (4.1)–(4.3) given by Vanden-Broeck (1981).

Graphs of $\theta(1)$ versus $\alpha^{-\frac{1}{2}}$ for the circular cylinder are shown in figure 6. The velocity at the separation points is infinite for $\gamma < \gamma^*$ and equal to zero for $\gamma > \gamma^*$.

Our results can be interpreted by introducing the contact angle β at the separation point A by the relation

$$\beta = \delta - \pi. \tag{4.14}$$

It can be argued that this angle depends only on the surface tension σ and on the physical properties of the surface of the obstacle (Batchelor 1967, p. 66). The angle β can therefore be considered as given.

Substituting (4.9) into (4.14) we obtain

$$\beta = \frac{C}{2}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}. \tag{4.15}$$

Relation (4.15) and figure 5 shows that the position of the separation point is uniquely determined by specifying the angle β and the Weber number α.

The existence of solutions for which the flow leaves the obstacle tangentially can be established as follows.

As α tends to zero, the free surfaces must approach two horizontal straight lines. Therefore

$$\lim_{\alpha\to 0} \theta(1) = 0. \tag{4.16}$$

Provided that $\theta(1)$ is a continuous function of α, figure 6 and (4.14) imply the existence for each value of $\gamma^* < \gamma < 90°$ of one value of $0 < \alpha < \infty$ for which $\theta_1 = -\frac{1}{2}\pi + \gamma$. We describe this relation between α and γ by the function

$$\gamma = g(\alpha). \tag{4.17}$$

This result can be reformulated as follows. For each value of the Weber number α there exists an angular position $\gamma = g(\alpha)$ of the separation points for which the flow leaves the obstacle tangentially.

As α tends to zero the free surfaces tend to two horizontal straight lines. This solution leaves the cylinder tangentially only if $\gamma = 90°$. Therefore

$$\lim_{\alpha\to 0} g(\alpha) = 90°. \tag{4.18}$$

As α tends to infinity, the solution is described by the asymptotic solution (4.7) and (4.8). This solution leaves the obstacle tangentially only if $C = 0$ (see (4.9)). Therefore figure 6 implies

$$\lim_{\alpha\to \infty} g(\alpha) = \gamma^*. \tag{4.19}$$

Relation (4.17) shows that the family of solutions defined by (4.15) tends to the classical solution satisfying the Brillouin–Villat condition as $\alpha \to \infty$.

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